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Jacobi Matrix Differential Equation, Polynomial Solutions, and Their Properties

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Abstract—In this paper, Jacobi matrix polynomials are introduced, starting from the hypergeometric matrix function. The differential equation satisfied by them is presented. A Rodrigues' formula, orthogonality, and a three terms matrix recurrence relationship are then developed for Jacobi matrix polynomials. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Orthogonal matrix polynomials comprise an emerging field of study, with important results in both theory and applications continuing to appear in the literature. Generalization of the concept of orthogonality from the scalar case has been considered in different ways [1–5], and development of other extensions, such as a Rodrigues'-type formula [6], a second-order Sturm-Liouville differential equation [6], or a three-term recurrence [7,8], for example, continue to emerge as they are uncovered. Applications of matrix polynomials also grow, and active areas in recent literature include statistics, group representation theory [9], scattering theory [10], differential equations [11,12], Fourier series expansions [13], interpolation and quadrature [14,15], splines [16], and medical imaging [7].

In the scalar case, the Jacobi polynomials $\{P_n^{(a,b)}(x)\}_{n=0}^{\infty}$ form a two parameter family with $a > -1$ and $b > -1$. One way of defining these polynomials is by an explicit formula such as [17]

$$P_n^{(a,b)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+a}{n-k} \binom{n+b}{k} (x-1)^k (x+1)^{n-k}. \tag{1}$$

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Then, they are shown to satisfy a second-order differential equation

$$\left(1-x^2\right) y''(x)+[b-a-(a-b+2)\, x]\, y'(x)+n(n+a+b+1) y(x)=0, \tag{2}$$

and a Rodrigues-type formula

$$P_n^{(a,b)}(x)=\frac{(-1)^n}{n!2^n}(1-x)^{-a}(1+x)^{-b} \frac{d^n}{dx^n}\left[(1-x)^{a+n}(1+x)^{b+n}\right]. \tag{3}$$

Orthogonality,

$$\int_{-1}^1(1-x)^a(1+x)^b P_n^{(a,b)}(x) P_m^{(a,b)}(x)\, dx=0, \qquad n \neq m, \tag{4}$$

may be deduced from (2) (a self-adjoint form is helpful), then used to construct a three-term recurrence relation

$$\begin{gathered}
2n(n+a+b)(2n+a+b-2) P_n^{(a,b)}(x) \\
=(2n+a+b-1)\left[(2n+a+b)(2n+a+b-2) x+a^2-b^2\right] P_{n-1}^{(a,b)}(x) \\
-2(n+a-1)(n+b-1)(2n+a+b) P_{n-2}^{(a,b)}(x).
\end{gathered} \tag{5}$$

As well, one of the special cases of interest above ((1)–(5)) comes about with the choice $a=b=-1/2$, the Chebyshev polynomials $T_n(x)$.

The outline analogous to (1)–(5) above is followed in establishing structures for Jacobi matrix polynomials, below. It is well recognized in the field, however, that the noncommutativity of matrix multiplication usually results in development of matrix analogs that does not have the relative simplicity found in the scalar situation. It is important, of course, to ensure that matrix generalizations do, indeed, collapse to appropriate scalar cases when the matrices involved are taken to have order one. Scalar instances of the matrix Jacobi development below are noted throughout.

This paper, then, is concerned with matrix polynomials

$$P_n(x)=A_n x^n+A_{n-1} x^{n-1}+\cdots+A_1 x+A_0$$

in which the coefficients A_i are members of $\mathbb{C}^{r\times r}$, the space of real or complex matrices of order r, and x is a real number. $P_n(x)$ is of degree n if A_n is not the zero matrix; for orthogonal matrix polynomials, the leading coefficient, A_n, being nonsingular is important [5,8].

In Section 2, we summarize basic facts, and properties to be used, in the following sections—such as about the matrix gamma function or the matrix hypergeometric function. Section 3 provides the definition of the Jacobi matrix polynomials, $P_n^{(A,B)}(x)$, for parameter matrices A and B whose eigenvalues, z, all satisfy $\operatorname{Re}(z)>-1$. The section also includes development of a second-order matrix Sturm-Liouville differential equation which is satisfied by the $P_n^{(A,B)}(x)$. A Rodrigues'-type formula, then integral orthogonality with a premultiplier weight,

$$\int_{-1}^1(1-x)^A(1+x)^B P_n^{(A,B)}(x) P_m^{(A,B)}(x)\, dx=0, \qquad n \neq m,$$

are obtained in Section 4. A three-term recurrence

$$\Lambda_{n+1} P_{n+1}^{(A,B)}(x)=\left(xI-\left\{I+\Lambda_n\right\}\right) P_n^{(A,B)}(x)-\Lambda_{n-1} P_{n-1}^{(A,B)}(x)$$

(Λ_{n+1} nonsingular), along with explicit expressions for the coefficient matrices Λ_i, are given in Section 5. The concluding comments in Section 6 include confirmation that the choice $A=C-I$,

$B = -C$ yields the one matrix-parameter, Chebyshev matrix family $\{T_n(x,C)\}_{n=0}^{\infty}$ obtained recently in [6].

Throughout this paper, for a matrix $A \in \mathbb{C}^{r\times r}$ its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^r$, $\|y\|_2 = (y^H y)^{1/2}$ is the Euclidean norm of y. I and θ will denote the identity matrix and the null matrix in $\mathbb{C}^{r\times r}$, respectively.

2. BACKGROUND NOTATION, FACTS, AND PROPERTIES

There are some basic facts or properties, and some notation or terminology, used throughout the development in Sections 3–5. They are listed here for easy referral in the sequel as "facts" or "notation", respectively, and references are given where appropriate.

NOTATION 2.1. *For $A \in \mathbb{C}^{r\times r}$, the matrix version of the Pochhammer symbol (the shifted factorial) is*

$$(A)_n = A\,(A+I)\,(A+2I)\cdots(A+(n-1)I)\,, \qquad n \geq 1,$$

with $(A)_0 \equiv I$.

Note that if $A = -jI$, where j is a positive integer, then $(A)_n = \theta$ whenever $n > j$.

NOTATION 2.2. *For $k = 0,1,2,\dots$*

$$D^k\,(f(x)) = \frac{d^k}{dx^k}\,(f(x))\,.$$

FACT 2.1. (See [18, p. 558].) If $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{r\times r}$ for which $\sigma(A) \subset \Omega$, then

$$f(A)g(A) = g(A)f(A). \tag{6}$$

Hence, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$ also, and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{7}$$

FACT 2.2. For an arbitrary matrix $A \in \mathbb{C}^{r\times r}$,

$$D^k\left[t^{A+mI}\right] = (A+I)_m\left[(A+I)_{m-k}\right]^{-1} t^{A+(m-k)I}, \qquad k = 0,1,2,\dots.$$

FACT 2.3. (See [19, p. 336,556].) For $A \in \mathbb{C}^{r\times r}$, let $\alpha(A) = \max\{\mathrm{Re}(z);\ z \in \sigma(A)\}$. Then

$$\left\|e^{At}\right\| \leq e^{t\alpha(A)} \sum_{k=0}^{r-1} \frac{\left(\|A\|\sqrt{r}t\right)^k}{k!}, \qquad t \geq 0. \tag{8}$$

As indicated in the introduction, the two-norm for matrices is used throughout.

FACT 2.4. (See [20, p. 253].) The reciprocal scalar *Gamma* function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{r\times r}$, the Riesz-Dunford functional calculus [18] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence:

if $C \in \mathbb{C}^{r\times r}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$(C)_n = \Gamma(C+nI)\Gamma^{-1}(C). \tag{9}$$

FACT 2.5. (See [21, p. 209].) If P and Q are members of $\mathbb{C}^{r\times r}$ for which $PQ = QP$, and if, for all nonnegative integers n,

$$P+nI,\ Q+nI,\ \text{and } P+Q+nI \text{ are all invertible,}$$

then

$$\mathcal{B}(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q), \tag{10}$$

where $\mathcal{B}(P,Q)$ denotes the *Beta* matrix function [21] acting on the pair P, Q.

FACT 2.6. If A, B, and C are members of $\mathbb{C}^{r\times r}$ for which

$$C+nI \text{ is invertible for every integer } n\geq 0,$$

then the hypergeometric matrix function $F(A,B;C;z)$ is defined by

$$F(A,B;C;z) = \sum_{n\geq 0} \frac{(A)_n(B)_n\left[(C)_n\right]^{-1}}{n!} z^n. \tag{11}$$

It converges for $|z| < 1$ [21].

FACT 2.7. (See [22].) Consider the hypergeometric matrix differential equation

$$z(1-z)\frac{d^2W(z)}{dz^2} - zA\frac{dW(z)}{dz} + \frac{dW(z)}{dz}\left(C - z(B+I)\right) - AW(z)B = \theta, \qquad 0<|z|<1, \tag{12}$$

in which $A, B, C \in \mathbb{C}^{r\times r}$ are constant matrices satisfying $CB = BC$, $AC = CA$, and

$$C+nI \text{ is invertible for all integers } n\geq 0.$$

Then the general solution of (12) (in some $a < |z| < \rho < 1$) is

$$W(z) = W_1(z)P + W_2(z)Q, \tag{13}$$

where

$$\begin{aligned} W_1(z) &= F(A,B;C;z), \qquad \text{and} \\ W_2(z) &= F\left(A+I-C, B+I-C; 2I-C; z\right) z^{I-C}. \end{aligned}$$

Here, P and Q are arbitrary constant matrices in $\mathbb{C}^{r\times r}$, and F is the hypergeometric matrix function (11). In particular, the hypergeometric matrix function (11) is a solution of (12) under the above conditions on matrices A, B, and C; so is a constant postmultiple of it.

The above facts and notation/definition material will be used throughout the next three sections.

3. DEFINITION AND MATRIX DIFFERENTIAL EQUATION

The Jacobi matrix polynomials are defined in (15) below, then the second-order differential equation they satisfy is derived, as Theorem 3.1 of this section.

DEFINITION 3.1. *Let A and B be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions*

$$\mathrm{Re}(z) > -1, \quad \forall z \in \sigma(A), \qquad \text{and} \qquad \mathrm{Re}(z) > -1, \quad \forall z \in \sigma(B). \tag{14}$$

For any natural number $n \geq 0$, the n^{th} Jacobi matrix polynomial $P_n^{(A,B)}(x)$ is defined by

$$\begin{aligned} P_n^{(A,B)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n+k}}{2^k n!} \Gamma(A+B+(n+k+1)I) \\ \cdot\Gamma^{-1}(A+B+(n+1)I)\Gamma(B+(n+1)I)\Gamma^{-1}(B+(k+1)I)(1+x)^k. \end{aligned} \tag{15}$$

REMARK 3.1. Note that

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{n!} F\left(A+B+(n+1)I, -nI; B+I; \frac{1+x}{2}\right) \Gamma^{-1}(B+I)\Gamma(B+(n+1)I), \tag{16}$$

and that for the scalar case $r = 1$, taking $A = a$ and $B = b$, and $a > -1$, $b > -1$, the n^{th} polynomial $P_n^{(a,b)}(x)$ coincides with the classical scalar Jacobi polynomial, see [23].

THEOREM 3.1. *For each natural number $n \geq 0$, the Jacobi matrix polynomial $P_n^{(A,B)}(x)$ satisfies the matrix differential equation*

$$\begin{aligned} \left(1-x^2\right) Y''(x) + 2Y'(x)B - (A+B+x(A+B+2I))\, Y'(x) \\ +n(A+B+(n+1)I)Y(x) = \theta, \qquad -1 < x < 1. \end{aligned} \tag{17}$$

PROOF. Consider the matrix differential equation

$$z(1-z)W''(z) - z\alpha W'(z) + W'(z)\,(\gamma - z(\beta+I)) - \alpha W(z)\beta = \theta, \qquad 0 < |z| < 1, \tag{18}$$

where α, β, γ are matrices in $\mathbb{C}^{r\times r}$ for which $\gamma\beta = \beta\gamma$, and

$$(\gamma + nI) \text{ is invertible for all positive integer } n \geq 0. \tag{19}$$

$$F(\alpha,\beta;\gamma;z) = \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n \left[(\gamma)_n\right]^{-1}}{n!} z^n \tag{20}$$

is therefore a solution of (18) in $0 < |z| < 1$ [22]. Taking $z = (x+1)/2$, $\alpha = A+B+(n+1)I$, $\beta = -nI$, and $\gamma = B+I$,

$$\begin{aligned} F\left(A+B+(n+1)I, -nI; B+I; \frac{1+x}{2}\right) = (-1)^n n! P_n^{(A,B)}(x) \\ \cdot\Gamma^{-1}(B+(n+1)I)\,\Gamma(B+I), \end{aligned} \tag{21}$$

from Remark 3.1. Introduce the notation

$$W\left(\frac{1+x}{2}\right) = F\left(A+B+(n+1)I, -nI; B+I; \frac{1+x}{2}\right). \tag{22}$$

Applying the chain rule in (22),

$$\begin{aligned} W'\left(\frac{1+x}{2}\right) &= 2(-1)^n n! \frac{d}{dx}\left(P_n^{(A,B)}(x)\right) \Gamma^{-1}(B+(n+1)I)\,\Gamma(B+I), \qquad \text{and} \\ W''\left(\frac{1+x}{2}\right) &= 4(-1)^n n! \frac{d^2}{dx^2}\left(P_n^{(A,B)}(x)\right) \Gamma^{-1}(B+(n+1)I)\,\Gamma(B+I). \end{aligned} \tag{23}$$

Taking into account that

$$B+I-\left(\frac{1+x}{2}\right)(1-n)I = \frac{1}{2}\left(2B+(1+n-x-xn)I\right)$$

and that this term commutes with $\Gamma^{-1}(B+(n+1)I)\Gamma(B+I)$, substituting (22),(23) in (18) and postmultiplying by $((-1)^n/n!)\Gamma^{-1}(B+I)\Gamma(B+(n+1)I)$ yields

$$\begin{aligned} \left(1-x^2\right)\frac{d^2}{dx^2}\left(P_n^{(A,B)}(x)\right) - (A+B+x(A+B+2I))\,\frac{d}{dx}\left(P_n^{(A,B)}(x)\right) \\ +2\frac{d}{dx}\left(P_n^{(A,B)}(x)\right)B + n\,(A+B+(n+1)I)\,P_n^{(A,B)}(x) = \theta. \end{aligned} \tag{24}$$

Thus, $P_n^{(A,B)}(x)$, as given by (15), satisfies (17) in $-1 < x < 1$. ∎

COROLLARY 3.1. *For $n \geq 0$, $P_n^{(A,B)}(x)$ is a solution of the differential equation*

$$\begin{gathered} \frac{d}{dx}\left[(1+x)(1-x)^{A+B+I}Y'(x)\left(\frac{1+x}{1-x}\right)^{B}\right] \\ +n\left(A+B+(n+1)I\right)(1-x)^{A+B}Y(x)\left(\frac{1+x}{1-x}\right)^{B}=\theta, \end{gathered} \tag{25}$$

over $-1 < x < 1$.

PROOF. Premultiplying (17) by $(1-x)^{A+B}$ and postmultiplying it by $((1+x)/(1-x))^B$, then rearranging yields (25) for $-1 < x < 1$. ∎

REMARK 3.2. Taking $A = a$ and $B = b$, with $a > -1$ and $b > -1$, in (17) produces the scalar Jacobi differential equation [17].

4. RODRIGUES' FORMULA AND ORTHOGONALITY

Two more basic properties of the Jacobi matrix polynomials $\{P_n^{(A,B)}(x)\}_{n=0}^{\infty}$ are developed in this section. That they enjoy a Rodrigues formula is obtained as Theorem 4.1, through exploitation of their definition (15) in terms of the hypergeometric matrix function (11). Orthogonality as an integral over $[-1,1]$ with a premultiplier weight $(1-x)^A(1+x)^B$ is the subject of Theorem 4.2 in Section 4.2. Throughout Section 4, the notation and hypotheses maintained in Section 3 will be used: $P_n^{(A,B)}(x)$ is defined by (15), with the two, parameter matrices satisfying

$$\operatorname{Re}(z) > -1, \quad \text{for } z \in \sigma(A), \qquad \operatorname{Re}(z) > -1, \quad \text{for } z \in \sigma(B), \quad \text{and} \quad AB = BA. \tag{26}$$

The second condition ($AB = BA$) now introduced here ensures that the differential equation (17) indeed has the general solution (13), and as well, for later, that $P_n^{(A,B)}(x)$ and $P_m^{(A,B)}(x)$ commute. The following terminology will also be employed.

DEFINITION 4.1. *A matrix $A \in \mathbb{C}^{r\times r}$ is called positively stable [13] if*

$$\operatorname{Re}(z) > 0, \qquad \forall\, z \in \sigma(A).$$

4.1. Rodrigues' Formula

The following lemma, derived in [6], will be useful in the sequel.

LEMMA 4.1. *For D and $C \in \mathbb{C}^{r\times r}$, suppose that D is positively stable, $BC = CB$, and that*

$$C - D + kI \text{ and } C + kI \text{ are invertible for all nonnegative integers } k.$$

Then, for $|t| < 1$,

$$F(-nI, D; C; t) = (1-t)^n F\left(-nI, C-D; C; \frac{-t}{1-t}\right), \qquad n = 0,1,2,\dots. \tag{27}$$

In definition (15), of the Jacobi matrix polynomials, the hypergeometric matrix function

$$F\left(A+B+(n+1)I, -nI; B+I; \frac{1+x}{2}\right) \tag{28}$$

is used. One of its properties is that

$$F\left(A+B+(n+1)I, -nI; B+I; \frac{1+x}{2}\right) = F\left(-nI, A+B+(n+1)I; B+I; \frac{1+x}{2}\right).$$

$D \stackrel{d}{=} (A + B + (n+1)I)$ is positively stable, and let C denote $B + I$. Then $DC = CD$ and the other hypotheses of Lemma 4.1 are satisfied as well. Hence,

$$\begin{aligned} &F\left(-nI, A+B+(n+1)I; B+I; \frac{1+x}{2}\right) \\ &= \frac{(1-x)^n}{2^n} F\left(-nI, -(A+nI); B+I; \frac{x+1}{x-1}\right). \end{aligned} \tag{29}$$

Substituting this last into (16) gives

$$\begin{aligned} P_n^{(A,B)}(x) &= \frac{(-1)^n}{2^n n!} F\left(-nI, -(A+nI); B+I; \frac{x+1}{x-1}\right) \Gamma^{-1}(B+I)\Gamma(B+(n+1)I) \\ &= \frac{(-1)^n}{2^n n!}(1-x)^n \sum_{k=0}^{n} \frac{(-n)_k \left(-(A+nI)_k\right)\left((B+I)_k\right)^{-1}}{k!} \left(\frac{x+1}{x-1}\right)^k (B+I)_n. \end{aligned} \tag{30}$$

With

$$(-n)_k = \begin{cases} (-1)^k \dfrac{n!}{(n-k)!}, & k \le n, \\ 0, & k > n, \end{cases}$$

equation (30) can be put in the form

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \left(-(A+nI)_k\right)\left((B+I)_k\right)^{-1}(B+I)_n (1+x)^k (1-x)^{n-k}. \tag{31}$$

Now,

$$(-(A+nI)_k) = (-A-nI)_k = (-1)^k (A+nI)(A+(n-1)I)\cdots(A+(n-k+1)I),$$

and hence,

$$\begin{aligned} D^{(k)}\left[(1-x)^{(A+nI)}\right] &= (-1)^k (A+nI)(A+(n-1)I)\cdots(A+(n-k+1)I)(1-x)^{A+(n-k)I} \\ &= (-(A+nI)_k)(1-x)^A (1-x)^{n-k}. \end{aligned}$$

Moreover,

$$D^{(n-k)}\left[(1+x)^{(B+nI)}\right] = ((B+I)_k)^{-1}(B+I)_n (1+x)^B (1+x)^k,$$

so (31) can be written in the form

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B} \sum_{k=0}^{n} \binom{n}{k} D^{(k)}\left[(1-x)^{(A+nI)}\right] D^{(n-k)}\left[(1+x)^{(B+nI)}\right].$$

Finally, applying Leibniz' rule for differentiations of a product yields that

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B} D^{(n)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right].$$

This result is summarized as follows.

THEOREM 4.1. RODRIGUES' FORMULA. *Let A and $B \in \mathbb{C}^{r\times r}$ satisfy (26). Then the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ defined in (15) may be expressed as*

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B} D^{(n)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right],$$

for $n = 0, 1, 2, \ldots$.

4.2. Orthogonality

Integral orthogonality over $-1 \le x \le 1$, with the premultiplying weight $W(x) = (1-x)^A(1+x)^B$, will now be derived for Jacobi matrix polynomials $P_n^{(A,B)}(x)$ having parameter matrices A and B which satisfy (26). A self-adjoint form of the Jacobi matrix differential equation, (25), will be useful, and, with it, behavior of certain matrix algebraic expressions involving $(1-x)^A(1+x)^B$, as $x \to \pm 1$, will be important. The first result of use in the development below is as follows.

LEMMA 4.2. *Let A and B be members of $\mathbb{C}^{r\times r}$ which satisfy (26), and let $Q(x)$ be an arbitrary matrix polynomial. Then*

$$\lim_{x\to 1-} \left(1-x^2\right)(1-x)^A(1+x)^B Q(x) = \theta, \qquad \textit{and}$$
$$\lim_{x\to -1+} \left(1-x^2\right)(1-x)^A(1+x)^B Q(x) = \theta.$$

PROOF. Observe that

$$\left(1-x^2\right)(1-x)^A(1+x)^B Q(x) = (1-x)^{A+I}(1+x)^{B+I}Q(x).$$

We consider the case $x \to 1-$ first. Let V be an open bounded neighborhood of $x=1$; $Q(x)$ is continuous, and hence, bounded on the closure of V, so

$$\|Q(x)\| \le K_1, \qquad \forall x \in V, \quad K_1 \in \mathbb{R}^+.$$

Recall that for an arbitrary $P \in \mathbb{C}^{r\times r}$, if $S^H PS = D + N$ is its Schur decomposition, and if

$$\alpha(P) = \max\left\{\operatorname{Re}(z);\ z \in \sigma(P)\right\},$$

then from (8)

$$\left\|e^{Pt}\right\| \le e^{t\alpha(P)} \left\{\sum_{k=0}^{r-1} \frac{\left(\|N\|\sqrt{r}t\right)^k}{k!}\right\}. \tag{32}$$

Hence,

$$\left\|(1-x)^{A+I}\right\| \le (1-x)^{\alpha(A+I)} \left\{\sum_{k=0}^{r-1} \frac{\left(\|N\|\sqrt{r}\log(1-x)\right)^k}{k!}\right\}, \tag{33}$$

where $Q^H(A+I)Q = D+N$ is the Schur decomposition of the matrix $A+I$. Since A satisfies (26), $A+I$ satisfies

$$\operatorname{Re}(z) > 0, \qquad \forall z \in \sigma(A+I),$$

so $\alpha(A+I) > 0$. Now, for $0 \le j \le r-1$,

$$\lim_{t\to 0+} t^a \left|\log(t)\right|^j = 0, \qquad a > 0,$$

and therefore

$$\lim_{x\to 1-} (1-x)^{\alpha(A+I)} \left|\log(1-x)\right|^j = 0, \qquad j = 0, 1, \dots, r-1.$$

But then

$$0 \le \left\|(1-x)^{A+I}\right\| \le (1-x)^{\alpha(A+I)} \left\{\sum_{k=0}^{r-1} \frac{\left(\|N\|\sqrt{r}\log(1-x)\right)^k}{k!}\right\},$$

and since the upper bound in this last expression approaches zero as $x \to 1-$,

$$\lim_{x\to 1-} (1-x)^{A+I} = \theta.$$

For the other component,

$$\lim_{x \to 1-} (1+x)^{B+I} = 2^{B+I},$$

and, on V, $\|(1+x)^{B+I}\|$ is bounded. Consequently,

$$\begin{aligned} 0 &\le \left\|(1-x)^{A+I}(1+x)^{B+I}Q(x)\right\| \\ &\le \left\|(1-x)^{A+I}\right\| \left\|(1+x)^{B+I}\right\| \|Q(x)\| \\ &\le K_1 \left\|(1-x)^{A+I}\right\| \left\|(1+x)^{B+I}\right\|, \end{aligned}$$

whence

$$\lim_{x \to 1-} (1-x)^{A+I}(1+x)^{B+I}Q(x) = \theta.$$

In a manner completely analogous to the above, it may be shown that

$$\lim_{x \to -1+} (1-x)^2(1-x)^A(1+x)^B Q(x) = \theta. \qquad \blacksquare$$

For the second stage in the derivation of orthogonality, observe that if the conditions for Theorem 4.1 hold, then the self-adjoint form (25) of equation (17) may be written in the form

$$\begin{aligned} &\frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_n^{(A,B)}(x)\right] \\ &+ n\left(A+B+(n+1)I\right)(1-x)^A(1+x)^B P_n^{(A,B)}(x) = \theta, \qquad -1 < x < 1. \end{aligned} \tag{34}$$

In this last, the fact that $P_n^{(A,B)}(x)$ is a solution of (25) has been utilized.

Multiplying (34) by $P_m^{(A,B)}(x)$ gives

$$\begin{aligned} &\frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_n^{(A,B)}(x)\right] P_m^{(A,B)}(x) \\ &+ n\left(A+B+(n+1)I\right)(1-x)^A(1+x)^B P_n^{(A,B)}(x) P_m^{(A,B)}(x) = \theta \end{aligned} \tag{35}$$

and, after exchanging m and n,

$$\begin{aligned} &\frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_m^{(A,B)}(x)\right] P_n^{(A,B)}(x) \\ &+ m\left(A+B+(n+1)I\right)(1-x)^A(1+x)^B P_m^{(A,B)}(x) P_n^{(A,B)}(x) = \theta. \end{aligned} \tag{36}$$

Subtracting (35) and (36), and using the fact that Jacobi polynomials commute,

$$\begin{aligned} &\frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_n^{(A,B)}(x)\right] P_m^{(A,B)}(x) \\ &- \frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_m^{(A,B)}(x)\right] P_n^{(A,B)}(x) \\ &+ (n-m)\left(A+B+(n+m+1)I\right) \\ &\cdot (1-x)^A(1+x)^B P_n^{(A,B)}(x) P_m^{(A,B)}(x) = \theta, \qquad -1 < x < 1. \end{aligned} \tag{37}$$

Now, it is easy to confirm that

$$\begin{aligned} &\frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \left\{P_m^{(A,B)}(x)\frac{d}{dx}P_n^{(A,B)}(x) - \frac{d}{dx}P_m^{(A,B)}(x)P_n^{(A,B)}(x)\right\}\right] \\ &= \frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_n^{(A,B)}(x)\right] P_m^{(A,B)}(x) \\ &- \frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B \frac{d}{dx} P_m^{(A,B)}(x)\right] P_n^{(A,B)}(x), \end{aligned} \tag{38}$$

and thus (37) may be rewritten as

$$\begin{aligned}&\frac{d}{dx}\left[\left(1-x^2\right)(1-x)^A(1+x)^B\left\{P_m^{(A,B)}(x)\frac{d}{dx}P_n^{(A,B)}(x)-\frac{d}{dx}P_m^{(A,B)}(x)P_n^{(A,B)}(x)\right\}\right]\\&\quad+(n-m)\left(A+B+(n+m+1)I\right)(1-x)^A(1+x)^B P_n^{(A,B)}(x)P_m^{(A,B)}(x)=\theta.\end{aligned}\tag{39}$$

Introduce the notation $Q(x)=P_m^{(A,B)}(x)\frac{d}{dx}P_n^{(A,B)}(x)-\frac{d}{dx}P_m^{(A,B)}(x)P_n^{(A,B)}(x)$, then integrate throughout (39) over $-1\le x\le 1$,

$$\begin{aligned}&(n-m)\left(A+B+(n+m+1)I\right)\int_{-1}^{1}(1-x)^A(1+x)^B P_n^{(A,B)}(x)P_m^{(A,B)}(x)\,dx\\&=\lim_{x\to 1-}\left(1-x^2\right)(1-x)^A(1+x)^B Q(x)-\lim_{x\to -1+}\left(1-x^2\right)(1-x)^A(1+x)^B Q(x).\end{aligned}\tag{40}$$

With application of Lemma 4.2, (40) implies that

$$\int_{-1}^{1}(1-x)^A(1+x)^B P_n^{(A,B)}(x)P_m^{(A,B)}(x)\,dx=\theta,$$

since $n\neq m$ and $(A+B+(n+m+1)I)$ is invertible.

For the final stage in the derivation of orthogonality, it is necessary to confirm that

$$\int_{-1}^{1}(1-x)^A(1+x)^B\left[P_n^{(A,B)}(x)\right]^2 dx$$

is invertible, for $n=0,1,2,\dots$.

From Rodrigues' formula (Theorem 4.1)

$$\begin{aligned}&\int_{-1}^{1}(1-x)^A(1+x)^B\left[P_n^{(A,B)}(x)\right]^2 dx\\&=\frac{(-1)^n}{n!}\int_{-1}^{1}P_n^{(A,B)}(x)D^{(n)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]dx.\end{aligned}\tag{41}$$

Observe that, by Lemma 4.2, for $n\ge 1$ and $1\le k\le n$, the following all hold:

$$\begin{aligned}&\lim_{x\to 1-}D^{(n-1)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)=\theta,\\&\lim_{x\to -1+}D^{(n-1)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)=\theta,\\&\lim_{x\to 1-}D^{(n-2)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)=\theta,\\&\lim_{x\to -1+}D^{(n-2)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)=\theta,\\&\qquad\vdots\qquad\text{and}\\&\lim_{x\to 1-}D^{(n-k)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)=\theta,\\&\lim_{x\to -1+}D^{(n-k)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)=\theta.\end{aligned}\tag{42}$$

A first application of integration by parts on the right side of (41) produces

$$\begin{aligned}&\frac{(-1)^n}{n!}\int_{-1}^{1}P_n^{(A,B)}(x)D^{(n)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]dx\\&=-\frac{(-1)^n}{n!}\left[\int_{-1}^{1}\frac{d}{dx}P_n^{(A,B)}(x)D^{(n-1)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]\right]dx\\&\quad+\frac{(-1)^n}{n!}\left[D^{(n-1)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]P_n^{(A,B)}(x)\right]_{-1}^{1},\end{aligned}$$

and from use of (42),

$$\frac{(-1)^n}{n!}\int_{-1}^{1} P_n^{(A,B)}(x)D^{(n)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]dx$$
$$=\frac{(-1)^{n+1}}{n!}\int_{-1}^{1}\frac{d}{dx}P_n^{(A,B)}(x)D^{(n-1)}\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right]dx.$$

Repeating this process $n-1$ times more, employing (42) each time, shows that

$$\int_{-1}^{1}(1-x)^A(1+x)^B\left[P_n^{(A,B)}(x)\right]^2 dx=\frac{1}{n!}\int_{-1}^{1}\frac{d^n}{dx^n}P_n^{(A,B)}(x)(1-x)^{(A+nI)}(1+x)^{(B+nI)}\,dx. \quad (43)$$

Directly from definition (15), however,

$$\frac{d^n}{dx^n}P_n^{(A,B)}(x)=\frac{1}{2^n}\Gamma(A+B+(2n+1))\Gamma^{-1}(A+B+(n+1)I),$$

and hence, (43) becomes

$$\begin{aligned}&\int_{-1}^{1}(1-x)^A(1+x)^B\left[P_n^{(A,B)}(x)\right]^2 dx\\ &=\frac{1}{2^n n!}\Gamma(A+B+(2n+1))\Gamma^{-1}(A+B+(n+1)I)\int_{-1}^{1}(1-x)^{(A+nI)}(1+x)^{(B+nI)}\,dx.\end{aligned} \quad (44)$$

To complete the evaluation of (44),

$$\int_{-1}^{1}(1-x)^{(A+nI)}(1+x)^{(B+nI)}\,dx=\int_{-1}^{1}(1+x)^{(B+nI)}(1-x)^{(A+nI)}\,dx$$

may be obtained in a closed form using the following result [6].

LEMMA 4.3. *Let D and F in $\mathbb{C}^{r\times r}$ satisfy*

$$\operatorname{Re}(z)>-1 \quad \text{and} \quad \operatorname{Re}(w)>-1, \qquad \text{for all } z\in\sigma(D) \text{ and all } w\in\sigma(F).$$

Then

$$\int_{-1}^{1}(1+x)^D(1-x)^F\,dx=2^{D+I}\mathcal{B}(D+I,F+I)\,2^F.$$

In Lemma 4.3, $\mathcal{B}$ denotes the Beta matrix function (10).

Putting $D=B+nI$ and $F=A+nI$, D and F satisfy the conditions of Lemma 4.3, and consequently

$$\int_{-1}^{1}(1+x)^{(B+nI)}(1-x)^{(A+nI)}\,dx=2^{B+(n+1)I}\mathcal{B}(B+(n+1)I,A+(n+1)I)\,2^{A+nI}.$$

Thus, by (10) it follows that

$$\mathcal{B}(B+(n+1)I,A+(n+1)I)=\Gamma(B+(n+1)I)\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+(2n+2)I).$$

Finally, substituting this last result into (44) shows that

$$\begin{aligned}\int_{-1}^{1}(1-x)^A(1+x)^B\left[P_n^{(A,B)}(x)\right]^2 dx&=2^{A+B+I}\frac{1}{n!}\Gamma(A+B+(2n+1))\\ &\cdot\Gamma^{-1}(A+B+(n+1)I)\Gamma(B+(n+1)I)\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+(2n+2)I),\end{aligned}$$

and hence, $\int_{-1}^{1}(1-x)^A(1+x)^B[P_n^{(A,B)}(x)]^2\,dx$ is nonsingular. In summary, we have obtained the following important result, orthogonality of the Jacobi matrix polynomials, $P_n^{(A,B)}(x)$, that are defined by (15).

THEOREM 4.2. *Let A and B in $\mathbb{C}^{r\times r}$ satisfy (26). Then for any nonnegative integers n and m,*

$$\int_{-1}^{1}(1-x)^A(1+x)^B P_n^{(A,B)}(x)P_m^{(A,B)}(x)\,dx$$
$$= \begin{cases} \theta, & n\neq m,\\ \dfrac{2^{A+B+I}}{n!}\Gamma(A+B+(2n+1))\Gamma^{-1}(A+B+(n+1)I) & \\ \cdot\Gamma(B+(n+1)I)\cdot\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+(2n+2)I), & n=m. \end{cases}$$

REMARK 4.1. By Theorem 4.2, the Jacobi matrix polynomial family $\{P_n^{(A,B)}(x)\}_{n=0}^{\infty}$ is orthogonal, in the sense defined in [11,14], with respect to the matrix weight $W(x,A,B)=(1-x)^A(1+x)^B$ over $[-1,1]$.

REMARK 4.2. In the scalar case of Theorem 4.2, the choice $A=a$ and $B=b$, with $a>-1$ and $b>-1$, yields the usual orthogonality for the classical scalar Jacobi polynomials $P_n^{(a,b)}(x)$ [17].

5. THREE-TERM RECURRENCE FORMULA

The last major property developed here is a three-term recurrence relation for the Jacobi matrix polynomials $\{P_n^{(A,B)}(x)\}_{n=0}^{\infty}$ derived from their orthogonality established by Theorem 4.2. As in earlier developments, the parameter matrices A and B will be assumed to satisfy conditions (26).

To begin, note first in definition (15) that the leading coefficient of $P_n^{(A,B)}(x)$ is an invertible matrix, for $n\geq 1$. By Theorem 2.2 of [5], any matrix polynomial of degree n can therefore be represented uniquely in the form

$$Q(x)=\sum_{k=0}^{n}\Lambda_k P_k^{(A,B)}(x),\qquad \text{for some } \Lambda_k\in\mathbb{C}^{r\times r}. \tag{45}$$

So, by Theorem 4.2 and (45), if $P(x)$ is a matrix polynomial of degree strictly less than n, then

$$\int_{-1}^{1}P(x)(1-x)^A(1+x)^B P_n^{(A,B)}(x)\,dx=\theta.$$

Now, $(1+x)P_n^{(A,B)}(x)$ is a matrix polynomial of degree $n+1$, for $n\geq 0$, and hence, from (45),

$$(1+x)P_n^{(A,B)}(x)=\sum_{k=0}^{n+1}\Lambda_k P_k^{(A,B)}(x), \tag{46}$$

for some coefficients Λ_k in $\mathbb{C}^{r\times r}$.

Using Theorem 4.2, and for $k=0,1,2,\dots,n+1$,

$$\int_{-1}^{1}(1+x)P_n^{(A,B)}(x)(1-x)^A(1+x)^B P_k^{(A,B)}(x)\,dx=\Lambda_k H_k; \tag{47}$$

the coefficient matrices Λ_k can be determined by Theorem 4.2. Consequently,

$$\int_{-1}^{1}(1+x)P_n^{(A,B)}(x)(1-x)^{A-I}(1+x)^B P_k^{(A,B)}(x)\,dx=\theta,\qquad \text{for } k+1<n,$$

so that

$$(1+x)P_n^{(A,B)}(x)=\Lambda_{n-1}P_{n-1}^{(A,B)}(x)+\Lambda_n P_n^{(A,B)}(x)+\Lambda_{n+1}P_{n+1}^{(A,B)}(x), \tag{48}$$

which is a three-term recurrence relation from the Jacobi matrix polynomials $P_n^{(A,B)}(x)$. Identifying equal powers of $(1+x)$ in (48), leads finally to explicit expressions for the recurrence coefficient matrices Λ_{n-1}, Λ_n, and Λ_{n+1}

$$\begin{aligned}
\Lambda_{n+1} &= 2(n+1)(A+B+(n+1)I)(A+B+(2n+1)I)^{-1}(A+B+(2n+2)I)^{-1}, \\
\Lambda_n &= -2n(B+nI)(A+B+2nI)^{-1} + 2(n+1)(A+B+(2n+2)I)^{-1}(B+(n+1)I), \\
\Lambda_{n-1} &= (n-1)(A+B+nI)^{-1}(B+nI)(B+(n-1)I) \\
&\quad - 2(n+1)(A+B+2nI)(A+B+(2n-1)I)(A+B+(2n+1)I)^{-1} \\
&\quad \cdot (A+B+(2n+2)I)^{-1}(A+B+nI)^{-1}(B+nI)(B+(n+1)I) \\
&\quad - 2n(A+B+(2n-1)I)(A+B+nI)^{-1}(B+nI)^2(A+B+2nI)^{-1} \\
&\quad + (A+B+(2n-1)I)(A+B+nI)^{-1}(B+nI) \\
&\quad \cdot 2(n+1)(A+B+(2n+2)I)^{-1}(B+(n+1)I).
\end{aligned} \tag{49}$$

These results are summarized below.

THEOREM 5.1. *Let A and B in $\mathbb{C}^{r\times r}$ satisfy conditions (26). Then the Jacobi matrix polynomials $\{P_n^{(A,B)}(x)\}_{n=0}^{\infty}$ defined in (15) satisfy the three-term matrix recurrence relation*

$$\begin{gathered}
P_{-1}^{(A,B)}(x) = \theta, \qquad P_0^{(A,B)}(x) = I, \\
\Lambda_{n+1} P_n^{(A,B)}(x) = (xI + \{I - \Lambda_n\})\, P_{n-1}^{(A,B)}(x) - \Lambda_{n-1} P_{n-1}^{(A,B)}(x),
\end{gathered} \tag{50}$$

in which Λ_{n-1}, Λ_n, and Λ_{n+1} in $\mathbb{C}^{r\times r}$ are given by (49). In (50), Λ_{n+1} is nonsingular for $n \geq 0$.

REMARK 5.1. If the matrix order, r, is set to 1 throughout (49),(50), then this recurrence indeed collapses to that for the scalar case as provided in [17].

6. CONCLUDING COMMENTS

The material developed in Sections 3–5 provides several important properties of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ introduced in (15), under the spectral conditions (14) on the parameter matrices A and B. The $P_n^{(A,B)}(x)$ are first shown to satisfy the second-order differential equation (17). Then, with the added condition of commutativity, $AB = BA$, a Rodrigues' formula for the $P_n^{(A,B)}(x)$ is derived (Theorem 4.1), orthogonality in the sense provided by Theorem 4.2 is established and, finally, a three-term recurrence for the $P_n^{(A,B)}(x)$ is obtained, as Theorem 5.1.

Analogous to a special choice for parameters in the scalar case of the Jacobi polynomials, we have the following.

REMARK 6.1. If C in $\mathbb{C}^{r\times r}$ satisfies the spectral condition

$$0 < \mathrm{Re}(z) < 1, \qquad \forall\, z \in \sigma(C),$$

take $A = C - I$ and $B = -C$; thus, A and B satisfy conditions (26). Using the Rodrigues' formula, Theorem 4.1, then yields that

$$\begin{aligned}
P_n^{(C-I,-C)}(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-(C-I)}(1+x)^{C} D^{(n)}\left[(1-x)^{(C-I+nI)}(1+x)^{(-C+nI)}\right] \\
&= \frac{(C)_n}{n!} T_n(x, C),
\end{aligned}$$

where $T_n(x, C)$ is the Chebyshev matrix polynomial defined in [6].

The commutativity condition $AB = BA$ introduced at the beginning of Section 4 permits the development in Sections 4 and 5, including establishment of orthogonality of the $P_n^{(A,B)}(x)$ using the weight $W(x) = (1-x)^A(1+x)^B$—in particular,

$$\int_{-1}^{1} (1-x)^A (1+x)^B P_n^{(A,B)}(x) P_m^{(A,B)}(x)\, dx = \theta, \qquad n \neq m.$$

A simple illustration about AB being equal to BA, or not, is informative.

EXAMPLE 6.1. Let
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
A and B satisfy (14), but $AB \neq BA$. With definition (15),

- $P_0^{(A,B)}(x) = I,$
- $P_1^{(A,B)}(x) = \dfrac{1}{2}(A + B + 2I)\,x + \dfrac{1}{2}(A - B);$

we will see that
$$\int_{-1}^{1} (1-x)^A (1+x)^B P_0^{(A,B)}(x) P_1^{(A,B)}(x)\, dx \neq \theta.$$
In fact here,
$$(1-x)^A = \begin{pmatrix} 1-x & 0 \\ 0 & 1 \end{pmatrix}, \quad (1+x)^B = \begin{pmatrix} 1 & \log(1+x) \\ 0 & 1 \end{pmatrix}, \quad P_1^{(A,B)}(x) = \frac{1}{2}\begin{pmatrix} 3x+1 & x-1 \\ 0 & 2x \end{pmatrix}.$$
Then
$$\begin{aligned} &\int_{-1}^{1} (1-x)^A (1+x)^B P_0^{(A,B)}(x) P_1^{(A,B)}(x)\, dx \\ &= \frac{1}{2}\int_{-1}^{1} \begin{pmatrix} 1-x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \log(1+x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3x+1 & x-1 \\ 0 & 2x \end{pmatrix} dx \\ &= \begin{pmatrix} 0 & \frac{1}{9}(5 - 6\log(2)) \\ 0 & 0 \end{pmatrix} \neq \theta. \end{aligned}$$

That is, without $AB = BA$, the corresponding Jacobi matrix polynomials $P_n^{(A,B)}(x)$ defined in (15) might not satisfy the orthogonality as given by Theorem 4.2. This matrix case orthogonality specializes to the traditional orthogonality of the Jacobi polynomials, in the scalar case.

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